Professor E. C. Pickering informs me "that photographs have been obtained at the Harvard College Observatory of all the stars hitherto discovered whose spectra consist mainly of bright lines and are of the class discovered by Rayet. Part of these have been photographed at Cambridge, and the remainder in Peru." He states that they may be divided into three sub-classes, according to the characters of their fifteen bright lines. He says, further: "Photographs of the spectrum of planetary nebulæ have also been obtained. They resemble closely the spectra described above, except that the line 500 is strongly marked; 470 is seen in most of them, while the lines due to hydrogen are also bright."

It would seem that Professor Pickering's photographs do not permit him to distinguish the different positions of the bright blue band in some of these stars, for he gives for all the stars the same position, namely, λ 470.

We regret that the insufficiency of our instrumental means has left our examination of the spectra of these stars less complete than we could wish. Our observations appear to us, however, to be conclusive on the main object of our enquiry, namely, that the bright blue band in the three Wolf-Rayet stars in Cygnus, and in D.M. $+37°$ 3821, is not coincident with the blue band of the Bunsen flame.

V. "On Stokes's Current Function." By R. A. SAMPSON, Fellow of St. John's College, Cambridge. Communicated by Professor GREENHILL, F.R.S. Received November 24, 1890.

(Abstract.)

In Maxwell's 'Electricity and Magnetism,'* a view is put forward, in accordance with which we may regard any irrotational motion in a perfect liquid, for which the velocity potential is a solid zonal harmonic, as due to the juxtaposition at the origin, and upon the axis of symmetry, of sinks and sources.

But, in a liquid, any irrotational motion which is symmetrical with respect to an axis gives a velocity potential which may be expressed as a sum of a series of solid zonal harmonics, their common axis being the axis of symmetry, and their origin arbitrary, provided

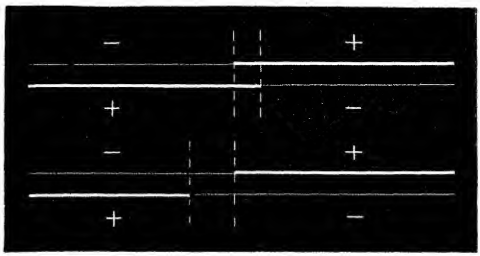
multiple image of four or five stars is surrounded by five bright and seven faint stars; all within a radial distance of 82″ of arc measured from centre to centre of the multiple star. The multiple image measures ±55″ in length and ±19″ in breadth.

"*No.* 3956.—Its photo-image is ±27″ in diameter. It is encircled by three stars of lesser brightness, and six faint ones within a radial distance of 59″, *i.e.*, there are nine stars within a radial distance of 59″."—Dec. 5.]

* Vol. 1, chapter ix.

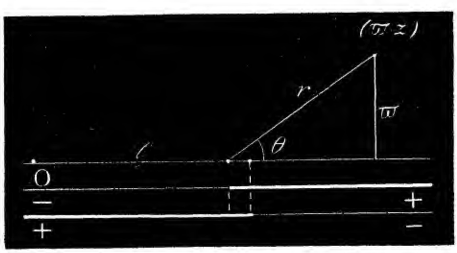
it is excluded from the region to which the expressions apply. The position of the origin upon the axis is arbitrary, since by a transference formula we may pass from one origin to another.

Let us now consider the system formed by a line source and a line sink, of equal strengths, extending along the axis from an arbitrary origin to infinity in opposite directions. Such a system I shall call an *extended doublet*, of strength m, where m is the strength per unit length of that part which lies on the positive side of the origin.



By the superposition of two extended doublets, of equal but opposite strengths, we can produce a sink or a source upon the axis. Hence, in a liquid, any irrotational motion which is symmetrical with respect to an axis, may be produced by superposition of extended doublets, whose origins depart but little from an arbitrary point on the axis of symmetry.

Now for an extended doublet of strength m, Stokes's current function ψ, for any point distant r from the the origin, is $-2mr$. For let ζ be the distance of the origin of the doublet from the origin of coordinates, and let $\psi(m, \zeta)$ be the value of Stokes's current function for



any point (ϖ, z). Then if $\delta\psi$ be the current function for a source of strength $2m\delta\zeta$, at the point ζ of the axis, we get

$$\frac{1}{\varpi}\frac{d}{dr}\cdot\delta\psi = 0.$$

$$\frac{1}{\varpi}\cdot\frac{d}{rd\theta}\cdot\delta\psi = -\frac{2m\delta\zeta}{r^2}.$$

Therefore $$\frac{d}{d\theta} \cdot \delta\psi = -2m \cdot \frac{\varpi}{r} \cdot \delta\zeta$$
$$= -2m \cdot \sin\theta \cdot \delta\zeta.$$

Whence $$\delta\psi = 2m\delta\zeta \cdot \cos\theta,$$

disregarding a constant.

But $$\delta\psi = \psi(m, \zeta) + \psi(-m, \zeta+\delta\zeta),$$

and clearly, $$\psi(-m, \zeta+\delta\zeta) = -\psi(m, \zeta+\delta\zeta).$$

Hence $$\psi(m, \zeta) - \left[\psi(m, \zeta) + \frac{d\psi}{d\zeta}\,\delta\zeta\right]$$
$$= \frac{d\psi(m, \zeta)}{d\zeta}\,\delta\zeta$$
$$= 2m\delta\zeta \cdot \cos\theta$$
$$= 2m\delta\zeta \cdot \frac{z-\zeta}{\sqrt{\{\varpi^2+(z-\zeta)^2\}}}.$$

Therefore $$\frac{d\psi}{d\zeta} = 2m \cdot \frac{z-\zeta}{\sqrt{\{\varpi^2+(z-\zeta)^2\}}},$$

and $$\psi = -2mr \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where $$r = \sqrt{(\varpi^2+\overline{z-\zeta}^2)},$$

disregarding a constant.

Thus if $m = f(\zeta)\,d\zeta$, we may, by properly choosing the function f, write

$$\psi = \int_{-\infty}^{+\infty} f(\zeta)\,\sqrt{\{\varpi^2+(z-\zeta)^2\}}\,d\zeta \quad\ldots\ldots\ldots\ldots \quad (2),$$

where ψ is the current function for any irrotational motion in a liquid, symmetrical about the axis of z.

Again, if $$r = \sqrt{\{\varpi^2+(z-\zeta)^2\}},$$

$$\frac{dr}{d\varpi} = \frac{\varpi}{r}, \qquad \frac{dr}{dz} = \frac{z-\zeta}{r},$$

$$\frac{d^2r}{d\varpi^2} = \frac{1}{r} - \frac{\varpi^2}{r^3}, \qquad \frac{d^2r}{dz^2} = \frac{1}{r} - \frac{(z-\zeta)^2}{r^3}.$$

Therefore $$\frac{d^2r}{d\varpi^2} + \frac{d^2r}{dz^2} = \frac{2}{r} - \frac{\varpi^2+(z-\zeta)^2}{r^3}$$
$$= \frac{1}{r}$$
$$= \frac{1}{\varpi}\,\frac{dr}{d\varpi},$$

and the expression (1), and consequently also (2), satisfies the differential equation

$$\frac{d^2\psi}{d\varpi^2}+\frac{d^2\psi}{dz^2}-\frac{1}{\varpi}\frac{d\psi}{dz}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

or, as I shall write it, $D\psi = 0$.

When the motion is rotational, (3) no longer holds. In fact, as is well known, we have under all circumstances

$$\frac{1}{\varpi}D\psi = -2\omega,$$

where ω is the resultant molecular rotation at the point (ϖ, z).

Thus, if there is molecular rotation in the fluid, (3) is replaced by

$$\frac{d^2\psi}{d\varpi^2}+\frac{d^2\psi}{dz^2}-\frac{1}{\varpi}\frac{d\psi}{dz} = -2\varpi\omega \quad \ldots\ldots\ldots\ldots \quad (3a).$$

Again, if ∇^2 stand for the operator $\frac{d^2}{d\varpi^2}+\frac{d^2}{dz^2}+\frac{1}{\varpi}\frac{d}{d\varpi}+\frac{1}{\varpi^2}\frac{d^2}{d\phi^2}$, ϕ being the azimuthal angle about the axis of symmetry, it may be seen at once that

$$D\psi = \frac{\varpi}{\sin\phi}\nabla^2\frac{\sin\phi}{\varpi}\psi \quad \ldots\ldots\ldots\ldots\ldots \quad (4).$$

Consequently (3a) may be written

$$\nabla^2\frac{\psi\sin\phi}{\varpi} = -2\varpi\omega\times\frac{\sin\phi}{\varpi}$$

$$= -2\omega\sin\phi \quad \ldots\ldots\ldots\ldots\ldots \quad (3a').$$

Consequently

$$\psi = \psi_0+\frac{\varpi}{2\pi\sin\phi}\iiint\frac{\sin\phi\,\omega\,dx\,dy\,dz}{r},$$

where ψ_0 is a solution of (3).

Or ψ consists of a solution of (3) together with $\frac{\varpi}{2\pi\sin\phi}\times$ the potential at the point considered of a distribution of mass of density at any point $\sin\phi\times$ the molecular rotation at that point. This result is given by Basset, 'Hydrodynamics,' vol. 2, § 306.

I give one other general result. Since

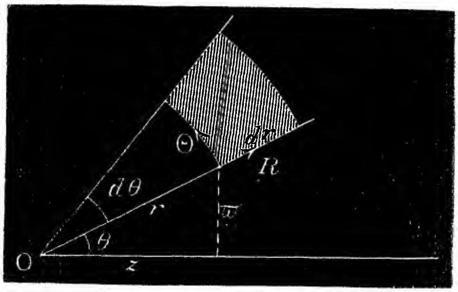
$$\omega = -\frac{1}{2\varpi}D\psi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3a),$$

the circulation in any evanescible circuit drawn in a meridional plane is

$$-\iint\frac{1}{\varpi}D\psi\,d\varpi\,dz \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

where the integration extends over the area embraced by the circuit.

This result enables us to transform $D\psi$ readily from cylindrical to other systems of coordinates. For instance, consider polar coordinates, r, θ, and let us find the circulation in a small rectangle bounded by r, $r+dr$, θ, $\theta+d\theta$.



Let the velocities in the direction of r and perpendicular to it be R, Θ. Then the circulation in this circuit is

$$\mathrm{R}\,dr+\left[\Theta r+\frac{d}{dr}(\Theta r)\,dr\right]d\theta-\left[\mathrm{R}+\frac{d\mathrm{R}}{d\theta}\,d\theta\right]dr-\Theta r\,d\theta$$

$$= r\,dr\,d\theta\left[\frac{d\Theta}{dr}+\frac{\Theta}{r}-\frac{d\mathrm{R}}{rd\theta}\right].$$

Now
$$\Theta=-\frac{1}{r\sin\theta}\cdot\frac{d\psi}{dr},$$

$$\mathrm{R}=\frac{1}{r^2\sin\theta}\cdot\frac{d\psi}{d\theta}.$$

Thus the expression in square brackets is—

$$-\frac{1}{r\sin\theta}\left[\frac{d^2\psi}{dr^2}+\frac{1}{r^2}\left(\frac{d^2\psi}{d\theta^2}-\cot\theta\frac{d\psi}{d\theta}\right)\right],$$

or
$$D\psi=\frac{d^2\psi}{dr^2}+\frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi}{d\theta}\right)$$

$$=\frac{d^2\psi}{dr^2}+\frac{1-\mu^2}{r^2}\frac{d^2\psi}{d\mu^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad(6),$$

if μ stands for $\cos\theta$.

Other applications will be found later.

Reverting now to the expression (1), it will be seen that the direct distance of any point from a point on the axis of symmetry plays the same part in the theory of Stokes's current function that is played by its reciprocal in the theory of the potential function belonging to symmetrical distributions of matter.

Thus if r_0, 0, r, θ, be the coordinates of a point upon the axis, and of any other point, the distance between these points, $\sqrt{(r_0^2-2r_0r\cos\theta+r^2)}$, may be developed in a convergent series, say

$$\sum_{n=0}^{n=\infty} -\frac{r^n}{r_0^{n-1}}\,\mathrm{I}_n(\cos\theta) \qquad \text{or} \qquad \sum_{n=0}^{n=\infty} -\frac{r_0^n}{r^{n-1}}\,\mathrm{I}_n(\cos\theta),$$

according as r_0 is greater or less than r, $\mathrm{I}_n(\cos\theta)$ being a certain function of θ, and we see from (6) that

$$(1-\mu^2)\frac{d^2\mathrm{I}_n(\mu)}{d\mu^2}+n\,(n-1)\,\mathrm{I}_n(\mu)=0 \;\ldots\ldots\ldots\; (7).$$

Now it is evident from the analogue of zonal harmonics that it is proper to discuss the function $\mathrm{I}_n(\cos\theta)$, and other solutions of (7) before considering the applications of Stokes's current function to the motion of liquids. It is with this discussion that the first three chapters are occupied, and, as might be expected, the theory closely resembles that of spherical harmonics. I have accordingly made free use of the order and methods adopted by Heine in his 'Handbuch d. Kugelfunctionen,' more especially in chapters i and ii,* where the necessary changes were slight. Moreover, the functions I deal with have themselves been discussed by Heine, on a different method, and most of the expressions which I find in the following pages are given by him. Full references to these are given in §18.

The idea of developing the solutions of $\mathrm{D}\psi=0$ in a manner more or less analogous to that employed with regard to Laplace's equation appears to have been first used by O. E. Meyer,† who obtains the equation (7), shows that the functions contain $1-\mu^2$ as a factor, and that they obey (28), chapter ii. An expression which shows the relation of the functions to zonal harmonics was given by Mr. Butcher;‡ and functions of fractional order have been used by Mr. Hicks,§ in connexion with his researches on the theory of the motion of vortex rings. The fuller account of such functions which is found in the following pages may be of interest in relation to these; for example, I would refer to § 63, chapter v.

* The following sections of the first three chapters contain methods or results which, so far as I am aware, are original:—12, 13, 17, 21, 25, 26, 29, 30, 32, 36, 38, 40, 42. The remainder of the paper is original, except where specially acknowledged, or where a result is too well known for that to be necessary.

† 'Crelle,' vol. 73, 1871.

‡ 'London Math. Soc. Proc.,' vol. 8. See p. 143, chapter vi.

§ 'Phil. Trans.,' 1884, 1885.

The applications to hydrodynamics which I here give are of mathematical interest rather than physical. They are chiefly in connexion with the motion of viscous liquids. In 'Crelle-Borchardt,' vol. 81, 1876, Oberbeck has given the velocities produced in an infinite viscous liquid by the steady motion of an ellipsoid through it, in the direction of one of its axes, and from these Mr. Herman* has found the equation of a family of surfaces containing the stream lines relative to the ellipsoid. In chapter vi, Stokes's current function is obtained by a direct process for the flux of a viscous liquid past a spheroid, and it is shown that the result differs only by a constant multiple from the particular case of Mr. Herman's integral.

Some minor applications are also given, namely, the solutions are obtained for flux past an approximate sphere, and past an approximate spheroid. The solution is also obtained for flux through a hyperboloid of one sheet, where it appears that the stream surfaces are hyperboloids of the confocal system. A particular case is that of flux through a circular hole in a wall, and this is interesting because we see that, by supposing internal friction to take place in the liquid, we find an expression which gives zero velocity at the sharp edge, and thus avoids the difficulty which is always present in the solution of such problems on the supposition that the liquid is perfect. A comparison may be instituted between this problem, and that of the effect of a disturbing periodic force upon a dynamical system capable of vibrating alone with a period equal to that of the force. It is well known that the amplitude of the vibration induced appears infinite, if we totally disregard friction, and this difficulty is met by the fact that the damping effect of even slight friction is rendered considerable by high velocities. Now a viscous liquid can move irrotationally, and, if there were no friction at the boundaries, this is the class of motion it would take in cases of flux past or through obstacles. But if the obstacle terminated in a sharp edge, this would make the velocity there infinite, and the friction, however inconsiderable elsewhere, would here become of account. The boundary conditions which were necessary for the existence of irrotational motion throughout the liquid would no longer apply, and the whole character of the solution would be changed. This would at any rate seem to apply to cases in which the whole motion is slow, and when, consequently, the boundary conditions which must hold are pretty well understood.

The paper concludes with an attempt to discuss the flux past a spheroid, or through a hyperboloid at whose boundary there may be slipping. The current function is not obtained, all that appears being that it probably differs from the parallel case of the sphere in being far more complicated than when there is no slipping. From

* 'Quart. Journ. Math.,' 1889 (No. 92).

this we except the case of the flux through a circular hole in a plane wall, when the solution for no slipping satisfies the new conditions.

Presents, December 11, 1890.

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